

ICERM Lecture 1

(Geodesic planes in ∞ -vol hyp mflds)

Hee Oh

(Yale University)

May 15, 2023

Geodesic planes in hyperbolic mflds of ∞ vol.

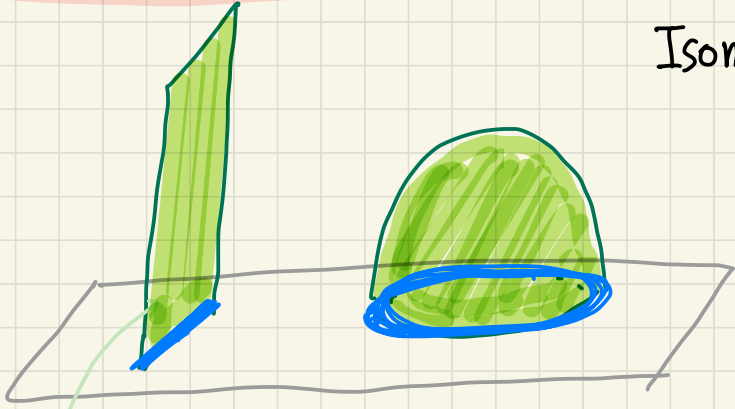
- Ref
- Geodesic planes in hyp 3-mflds
(McMullen-Mohammadi-O. - Inventiones 2017) 36 p.
 - Horocycles in hyp 3-mflds (MMO,
- GAFA 2016) 12 p.
 - Geodesic planes in the convex core of an
acylindrical 3-mfld (MMO
- Duke 2022) 31 p.
 - Geodesic planes in geom. finite
acylindrical 3-mflds (Benoist-O.
- ETDS 2022) 39 p.
 - Orbit closures of unip flows
for hyp mflds with Fuchsian ends
(Minju Lee-O. To appear G & T)
101 p.

* Dynamics for discrete subgps of $SL_2\mathbb{C}$
in Dynamics, Geometry, Number theory (Margulis vol)

$$ds = \frac{dx^2 + dy^2}{y}$$

geodesic planes in $\mathbb{H}^3 = \{(x_1, x_2, y) \mid y > 0\}$

$$\text{Isom}^+(\mathbb{H}^3) = \text{PSL}_2\mathbb{C} \cong \text{SO}^0(3,1)$$



$$\partial\mathbb{H}^3 = \mathbb{R}^2 \cup \{\infty\} = \mathbb{S}^2$$

$$\mathbb{H}^2 = \{(x, 0, y) \mid y > 0\}$$

$$\tilde{P} = g(\mathbb{H}^2) \quad g \in \text{PSL}_2\mathbb{C}$$

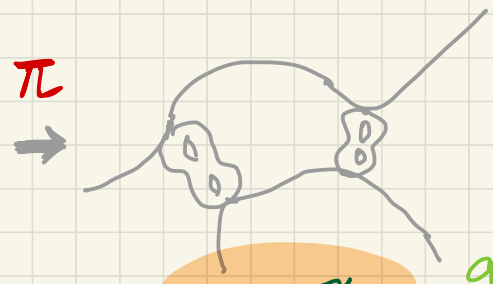
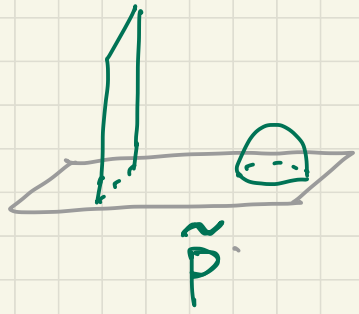
geodesic planes in \mathbb{H}^3

$$\partial\tilde{P} = g(\hat{\mathbb{R}})$$

circles in \mathbb{S}^2

$$M = \Gamma \backslash \mathbb{H}^3$$

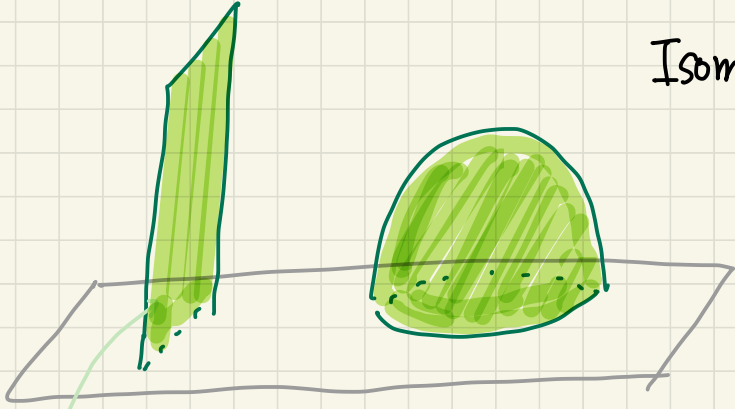
$\Gamma < \text{PSL}_2\mathbb{C}$ t.f discrete



$P = \pi(\tilde{P})$ geod. planes in M

geodesic k planes in $\mathbb{H}^n = \{(x_1, \dots, x_{n-1}, y) \mid y > 0\}$

$$\text{Isom}^+(\mathbb{H}^n) \simeq \text{SO}^0(n, 1) \cong G$$



$$\partial \mathbb{H}^n = \mathbb{R}^{n-1} \cup \{\infty\} = \mathbb{S}^{n-1}$$

$$\mathbb{H}^k = \{(x_1, \dots, x_k, 0, \dots, 0, y) \mid y > 0\}$$

$$\tilde{P} = g(\mathbb{H}^k) \quad g \in G$$

geodesic k planes in \mathbb{H}^n

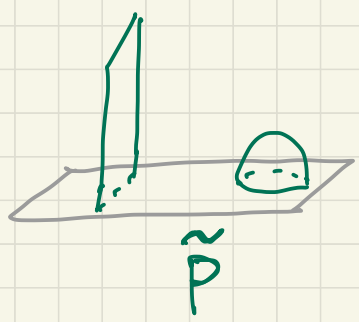
$$\partial \tilde{P} = g(\hat{\mathbb{R}}^{k-1})$$

$(k-1)$ spheres in \mathbb{S}^{n-1}

$$M = \Gamma \backslash \mathbb{H}^n$$

$$\Gamma < G$$

t.f discrete



geod. k planes in M

Question

P geod k -plane in $M = \Gamma \backslash \mathbb{H}^n$
 $2 \leq k \leq n-1$

Describe \overline{P} !

A Thm (Ratner, Shah ~ 90)

$M = \Gamma \backslash \mathbb{H}^n$ finite vol

$P \subset M$ geod k -plane $2 \leq k \leq n-1$

$\rightarrow \overline{P}$ is a geod m -plane $k \leq m \leq n$

(e.g. If $n=3$, $\overline{P} = P$ or $\overline{P} = M$)

What about $\text{vol}(M) = \infty$?

$M = \Gamma \backslash \mathbb{H}^n$ Convex cocompact

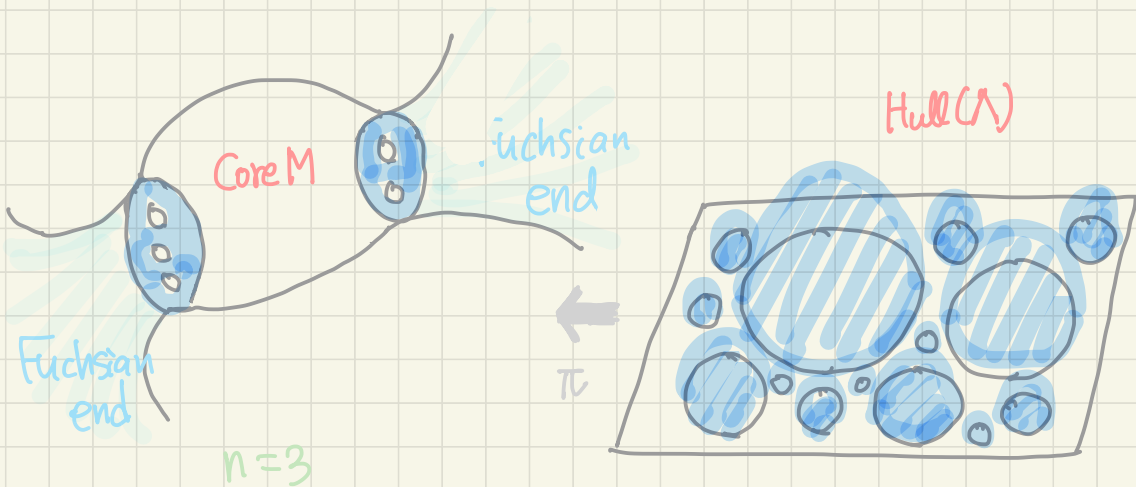
$\text{Int}(\text{Core } M) \neq \emptyset$ ($= \Gamma < \text{SO}^\circ(n,1) = G$)

Zariski dense

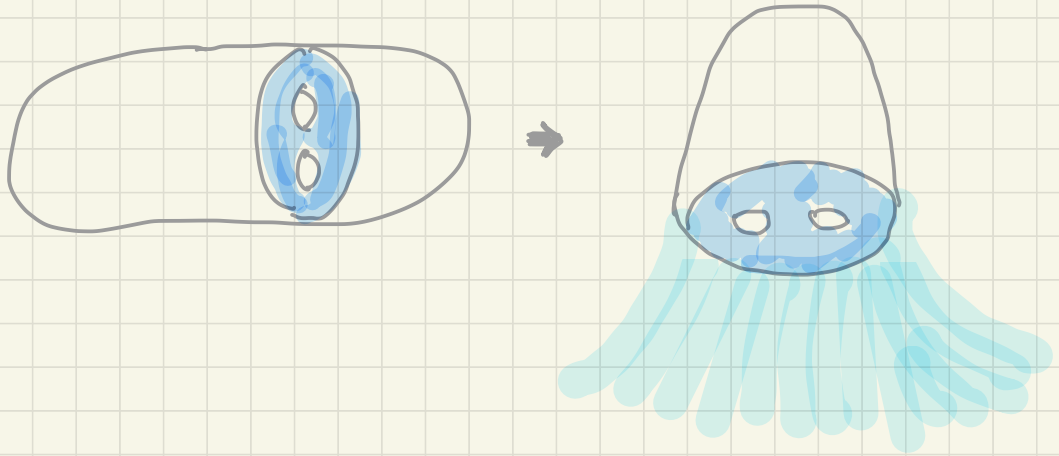
B Def M has Fuchsian ends

$= \partial(\text{Core } M)$ is totally geodesic

$= S^{n-1} - \Lambda = \bigcup_i B_i$ $\bar{B}_i \cap \bar{B}_j = \emptyset$
round balls



$\left\{ \begin{array}{l} \text{closed hyp} \\ n\text{-mfld} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{closed hyp } n\text{-mfld} \\ \text{with properly embedded} \\ \text{codim } 1 \text{ geod. planes} \end{array} \right\}$



Two types of planes:

$$\left\{ \begin{array}{l} P \cap \text{Int}(\text{Core } M) \neq \emptyset \\ P \cap \text{Int}(\text{Core } M) = \emptyset \end{array} \right.$$

$$P \cap \text{Int}(\text{Core } M) = \emptyset$$

$$\Rightarrow \bar{P} \subset M - \text{Int}(\text{Core } M) \quad \text{"Fuchsian ends"}$$

Can use Ratner-Shah to describe \bar{P} .

Thm (McMullen - Mohammadi - O. $n=3$
 Minju Lee - O. $n \geq 4$)

$M = \Gamma \backslash \mathbb{H}^n$: hyp mfd with Fuchsian ends

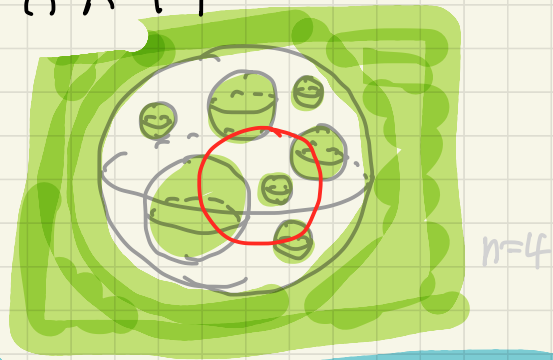
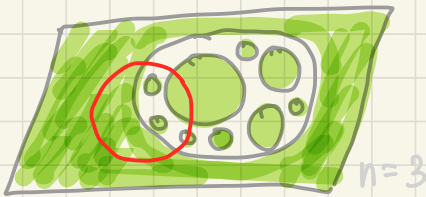
$P \subset M$ geod k -plane $2 \leq k \leq n-1$

$P \cap \text{Int}(\text{Core} M) \neq \emptyset$

$\rightarrow \bar{P} = \text{geod } m\text{-plane}$ $k \leq m \leq n$

For $n=3$, $\bar{P} = P$ or $\bar{P} = M$.

$$\mathbf{C}_\Lambda = \left\{ C \subset \mathbb{S}^{n-1} \mid \begin{array}{l} (k-1)\text{ sphere} \\ C \cap \Lambda \neq \emptyset \end{array} \right\}$$



$$C \in \mathbf{C}_\Lambda \quad |C \cap \Lambda| \geq 2$$

for some $(m-1)$ sphere

$$\bar{\Gamma}C = \{ D \in \mathbf{C}_\Lambda \mid D \subset \Gamma S \} \quad S \subset \mathbb{S}^{n-1}$$

For $n=3$, $\bar{\Gamma}C = \Gamma C$ or $\bar{\Gamma}C = \mathbf{C}_\Lambda$

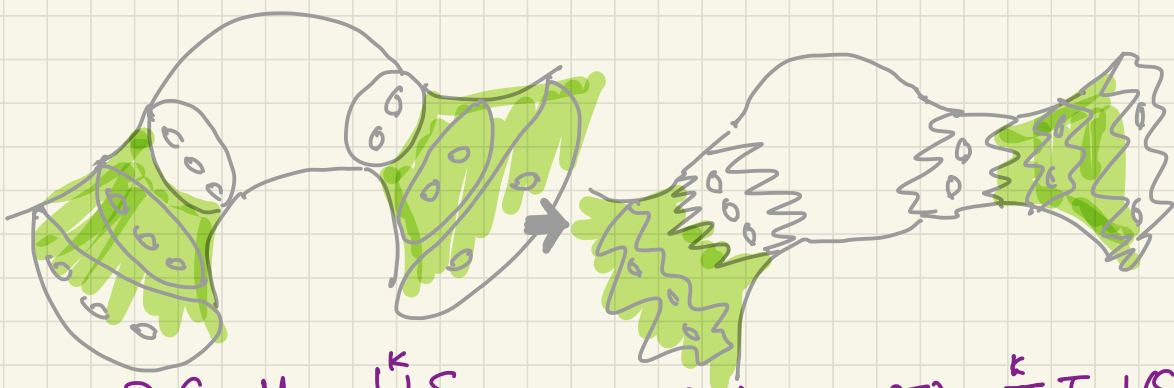
C $M = \mathbb{P}^1 \setminus \mathbb{H}^3$ Convex Cocpt

Def M has quasi-Fuchsian ends

$= M \underset{Q.I}{\sim} M_0 = \mathbb{H}^3 / \Gamma_0$ CC with Fuchsian ends

$= \Gamma$ is a Quasi-conf. def of Γ_0

$= \mathbb{S}^2 - \Lambda = \cup B_i \quad \bar{B}_i \cap \bar{B}_j = \emptyset$
 ↑ Jordan disks



$$\partial \text{Core } M_0 = \bigcup_{i=1}^k S_i$$

$$Q.I(M_0) = Q.C(\Gamma_0) = \prod_{i=1}^k \text{Teich}(S_i) \\ \simeq \prod \mathbb{R}^{6g_i - 6}$$



Thm (McMullen-Mohammadi-O.)

$M = \Gamma \backslash \mathbb{H}^3$ quasi-Fuchsian ends

$P \subset M$ geod plane with $P \cap \text{Int}(\text{Core } M) \neq \emptyset$

→ P is either closed or dense in $\text{Int}(\text{Core } M)$

i.e., $\overline{P} \cap \text{Int}(\text{Core } M) = \begin{cases} P \cap \text{Int}(\text{Core } M) \\ \text{Int}(\text{Core } M) \end{cases}$

Thm (Yongquan Zhang)

$\exists P \subset M$ s.t

P is closed in $\text{Int}(\text{Core } M)$

but not closed in M .

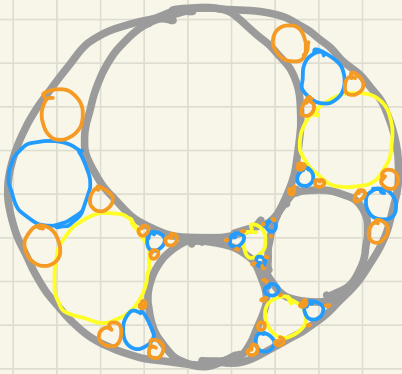
Rank \overline{P} for $P \subset \text{Ends of } M$

is not completely understood.

Rmk Analogous results for some
 geom. finite hyp 3-mflds
 were obtained by Benoist-O.

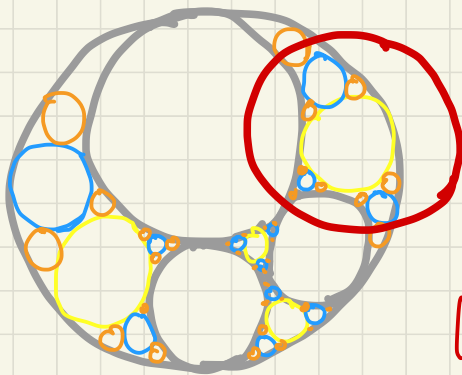
Open problem

\mathcal{P} :



Apollonian circle packing

$$\Gamma = \{ g \in \text{PSL}_2(\mathbb{C}) \mid g\mathcal{P} = \mathcal{P} \}$$



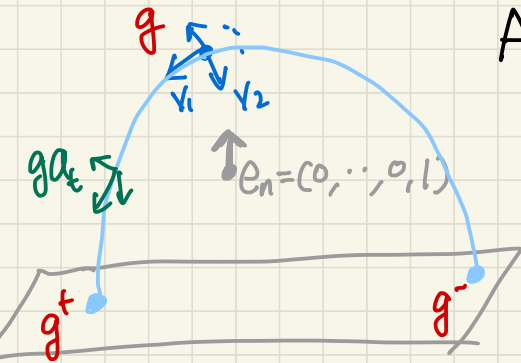
$$\overline{\Gamma \mathbb{C}} = \left\{ \begin{array}{l} \Gamma \mathbb{C} \\ \mathbb{C} \wedge \end{array} \right. \quad ?$$

$$|\mathbb{C} \wedge| \geq 2$$

Homogeneous dynamics.

$$G = \text{Isom}^+(\mathbb{H}^n) = \text{SO}^\circ(Q) \quad Q = 2x_1x_{n+1} + \sum_{i=2}^n x_i^2$$

$$A = \{a_t = \begin{pmatrix} e^t & & \\ & 1 \dots 1 & \\ & & e^{-t} \end{pmatrix} \mid t \in \mathbb{R}\}$$



$$F(\mathbb{H}^n) = \frac{\text{SO}(n,1)}{G} \leftarrow a_t \quad \text{Frame flow}$$

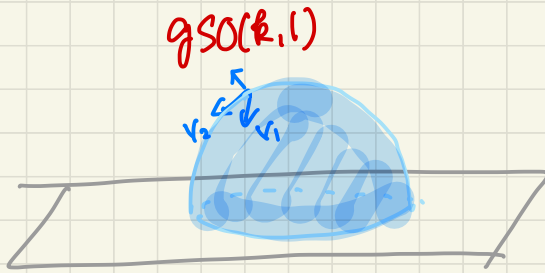
$$T^1(\mathbb{H}^n) = \frac{G}{\text{SO}(n-1)} \leftarrow a_t \quad \text{geodesic flow}$$

$$\mathbb{H}^n = \frac{G}{\text{SO}(n)}$$

$g\text{SO}(k,1)$: $\text{SO}(k,1)$ orbits



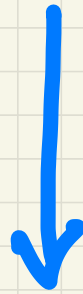
(oriented)
 $g\mathbb{H}^k$: geodesic k -planes



$$F(M) = \mathbb{P}^1 G \leftarrow a_t \text{ Frame flow}$$

$[g]SO(k,1)$: $SO(k,1)$ orbits

$$T^1(M) = \mathbb{P}^1 G / SO(n-1) \leftarrow a_t \text{ geodesic flow}$$



$$M = \mathbb{P}^1 G / SO(n)$$

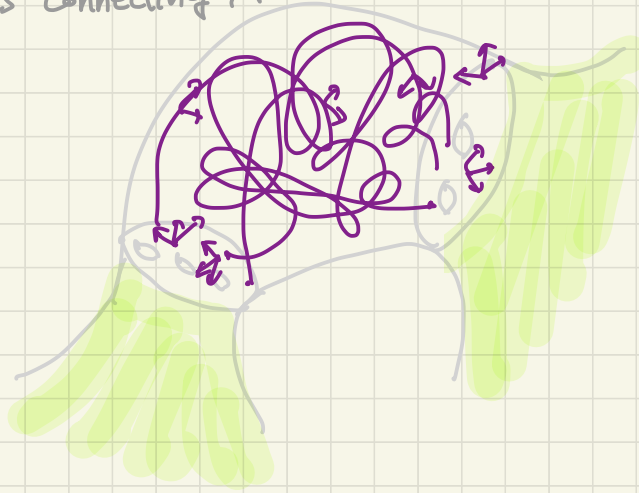
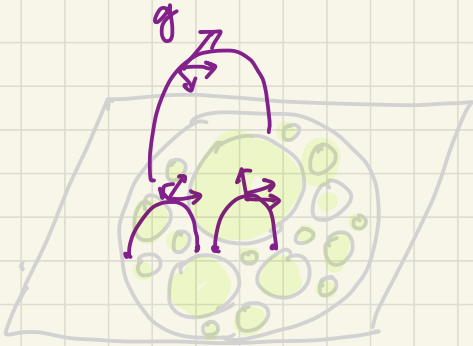
$\pi(gH^k)$: geodesic k -planes

Describe $[g]SO(k,1)$ in $\mathbb{P}^1 G$

$$\Omega = \text{RFM} = \left\{ [g] \in \frac{G}{\Gamma} \mid g^\pm \in \Lambda \right\}$$

opt A -inv
subset

union of all geodesics connecting Λ



Let $M = \frac{\mathbb{H}^n}{\Gamma}$ have Fuchsian ends

Thm (McMullen-Mohammadi-O. $n=3$ Lee-O. $n \geq 4$)

$W < G$ conn subgp gen. by unipotent elts
(\cong normalized by A)

Any W -orbit closure is relative homogeneous in Ω

$$\forall x \in \Omega, \overline{xW} \cap \Omega = xL \cap \Omega$$

where $W < L < G$ \cong xL closed.

Moreover, $2 \leq k \leq n-1$

$$\overline{xSO(k,1) \cap \Omega} = xSO(m,1)C \cap \Omega$$

where $C \subset$ Centralizer $SO(m,1)$
closed subgp $= SO(n-m)$

$$\overline{xSO(k,1)} = xSO(m,1)C \cap \Omega_+SO(k,1)$$

where $\Omega_+ = \{[g] \in \frac{G}{\Gamma} \mid g^+ \in \Lambda\}$

If M has empty Fuchsian ends (ie, M cpt),

$\Omega = \frac{G}{\Gamma}$ & this is Ratner & Shah.

Thank you !